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2008 J. Phys. A: Math. Theor. 41 025204

(http://iopscience.iop.org/1751-8121/41/2/025204)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **41** (2008) 025204 (12pp) doi:10.1088/1751-8113/41/2/025204

Closed conformal Killing–Yano tensor and geodesic integrability

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Received 5 September 2007, in final form 19 November 2007
Published 19 December 2007
Online at stacks.iop.org/JPhysA/41/025204

Abstract
Assuming the existence of a single rank-2 closed conformal Killing–Yano tensor with a certain symmetry we show that there exists mutually commuting rank-2 Killing tensors and Killing vectors. We also discuss the condition of separation of variables for the geodesic Hamilton–Jacobi equations.

PACS numbers: 02.40.Ky, 04.70.Bw

1. Introduction

Recently, it has been shown that the geodesic motion in the Kerr–NUT–de Sitter spacetime is integrable for all dimensions [1–6]. Indeed, the constants of motion that are in involution can be explicitly constructed from a rank-2 closed conformal Killing–Yano (CKY) tensor. In this paper, we answer the question raised in [5] under which general assumptions a CKY tensor implies the complete integrability of geodesic equation. We assume the existence of a single rank-2 closed CKY tensor with a certain symmetry for the D-dimensional spacetime M with a metric g. It turns out that such a spacetime admits mutually commuting k rank-2 Killing tensors and k Killing vectors. Here we put $D = 2k$ for even D, and $D = 2k - 1$ for odd D. Although the existence of the commuting Killing tensors was shown in [5, 6], we reproduce it more directly. We also discuss the condition of separation of variables for the geodesic Hamilton–Jacobi equations using the result given by Benenti–Francaviglia [7] and Kalnins–Miller [8] (see also [9]).

2. Assumptions and main results

A two-form

$$h = \tfrac{1}{2} h_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b, \qquad h_{ab} = -h_{ba} \tag{2.1}$$

is called a conformal Killing–Yano (CKY) tensor if it satisfies

$$\nabla_a h_{bc} + \nabla_b h_{ac} = 2\xi_c g_{ab} - \xi_a g_{bc} - \xi_b g_{ac}. \tag{2.2}$$

The vector field ξ_a is called the associated vector of h_{ab}, which is given by

$$\xi_a = \frac{1}{D-1}\nabla^b h_{ba}. \tag{2.3}$$

In the following we assume

$$(a1)\quad \mathrm{d}h = 0, \qquad (a2)\quad \mathcal{L}_\xi g = 0, \qquad (a3)\quad \mathcal{L}_\xi h = 0. \tag{2.4}$$

Assumption $(a1)$ means that $(D-2)$-form $f = *h$ is a Killing–Yano (KY) tensor,

$$\nabla_{(a_1} f_{a_2)a_3\cdots a_{D-1}} = 0. \tag{2.5}$$

Note that equation (2.2) together with $(a1)$ is equivalent to

$$\nabla_a h_{bc} = \xi_c g_{ab} - \xi_b g_{ac}. \tag{2.6}$$

It was shown in [10] that the associated vector ξ satisfies

$$\nabla_a \xi_b + \nabla_b \xi_a = \frac{1}{D-2}\left(R_a{}^c h_{bc} + R_b{}^c h_{ac}\right), \tag{2.7}$$

where R_{ab} is a Ricci tensor. If M is Einstein, i.e. $R_{ab} = \Lambda g_{ab}$, then

$$\nabla_a \xi_b + \nabla_b \xi_a = 0. \tag{2.8}$$

Thus, any Einstein space satisfies assumption $(a2)$ [10]. According to [5], we define $2j$-forms $h^{(j)}$ $(j = 0, \ldots, k-1)$:

$$h^{(j)} = \underbrace{h \wedge h \wedge \cdots \wedge h}_{j} = \frac{1}{(2j)!} h^{(j)}_{a_1\ldots a_{2j}} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_{2j}}, \tag{2.9}$$

where the components are written as

$$h^{(j)}_{a_1\ldots a_{2j}} = \frac{(2j)!}{2^j} h_{[a_1 a_2} h_{a_3 a_4} \cdots h_{a_{2j-1} a_{2j}]}. \tag{2.10}$$

Since the wedge product of the two CKY tensors is again a CKY tensor [5], $h^{(j)}$ are closed CKY tensors, and so $f^{(j)} = *h^{(j)}$ are KY tensors. Explicitly, we have

$$f^{(j)} = *h^{(j)} = \frac{1}{(D-2j)!} f^{(j)}_{a_1\ldots a_{D-2j}} \,\mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_{D-2j}}, \tag{2.11}$$

where

$$f^{(j)}_{a_1\ldots a_{D-2j}} = \frac{1}{(2j)!} \varepsilon^{b_1\ldots b_{2j}}{}_{a_1\ldots a_{D-2j}} h^{(j)}_{b_1\ldots b_{2j}}. \tag{2.12}$$

Given these KY tensors, we can construct the rank-2 Killing tensors $K^{(j)}$ obeying the equation $\nabla_{(a} K^{(j)}_{bc)} = 0$:

$$K^{(j)}_{ab} = \frac{1}{(D-2j-1)!(j!)^2} f^{(j)}_{ac_1\ldots c_{D-2j-1}} f_b^{(j)c_1\ldots c_{D-2j-1}}. \tag{2.13}$$

From $(a2)$ we have $\mathcal{L}_\xi *h^{(j)} = *\mathcal{L}_\xi h^{(j)}$ and hence assumption $(a3)$ yields

$$\mathcal{L}_\xi h^{(j)} = 0, \qquad \mathcal{L}_\xi f^{(j)} = 0, \qquad \mathcal{L}_\xi K^{(j)} = 0. \tag{2.14}$$

We also immediately obtain from (2.6)

$$\nabla_\xi h^{(j)} = 0, \qquad \nabla_\xi f^{(j)} = 0, \qquad \nabla_\xi K^{(j)} = 0. \tag{2.15}$$

Let us define the vector fields $\eta^{(j)}$ by [11, 12]

$$\eta_a^{(j)} = K^{(j)}{}_a{}^b \xi_b. \tag{2.16}$$

Then we have

$$\nabla_{(a}\eta_{b)}^{(j)} = \tfrac{1}{2}\mathcal{L}_\xi K_{ab}^{(j)} - \nabla_\xi K_{ab}^{(j)}, \tag{2.17}$$

which vanishes by (2.14) and (2.15), i.e. $\eta^{(j)}$ are Killing vectors.

Theorem 1 was proved in [5, 6].

Theorem 1. *Under* $(a1)$ *Killing tensors* $K^{(i)}$ *are mutually commuting,*

$$[K^{(i)}, K^{(j)}]_S = 0.$$

The bracket $[,]_S$ *represents a symmetric Schouten bracket. The equation can be written as*

$$K_{d(a}^{(i)}\nabla^d K_{bc)}^{(j)} - K_{d(a}^{(j)}\nabla^d K_{bc)}^{(i)} = 0. \tag{2.18}$$

Adding assumptions $(a2)$ *and* $(a3)$ *we prove*

Theorem 2.

$$\mathcal{L}_{\eta^{(i)}} h = 0.$$

Corollary. *Killing vectors* $\eta^{(i)}$ *and Killing tensors* $K^{(j)}$ *are mutually commuting,*

$$[\eta^{(i)}, K^{(j)}]_S = 0, \qquad [\eta^{(i)}, \eta^{(j)}] = 0.$$

3. Proof of theorems 1, 2

Let $H, Q := -H^2, K^{(j)}$ be matrices with elements

$$H^a{}_b = h^a{}_b, \qquad Q^a{}_b = -h^a{}_c h^c{}_b, \qquad (K^{(j)})^a{}_b = K^{(j)a}{}_b. \tag{3.1}$$

The generating function of $K^{(j)}$ can be read off from [5]

$$K_{ab}(\beta) = \sum_{j=0}^{k-1} K_{ab}^{(j)}\beta^j = \det{}^{1/2}(I + \beta Q)[(I + \beta Q)^{-1}]_{ab}. \tag{3.2}$$

Here $k = [(D + 1)/2]$. Note that

$$\begin{aligned} &2\det{}^{1/2}(I + \beta Q)[(I + \beta Q)^{-1}]^a{}_b \\ &\quad = \det(I + \sqrt{\beta}H)[(I + \sqrt{\beta}H)^{-1}]^a{}_b + \det(I - \sqrt{\beta}H)[(I - \sqrt{\beta}H)^{-1}]^a{}_b. \end{aligned} \tag{3.3}$$

Since $\det(I \pm \sqrt{\beta}H)[(I \pm \sqrt{\beta}H)^{-1}]^a{}_b$ is a cofactor of the matrix $I \pm \sqrt{\beta}H$, (3.2) is indeed a polynomial of β of degree $[(D - 1)/2]$.

For simplicity, let us define a matrix $S(\beta)$ by

$$S(\beta) := (I + \beta Q)^{-1}. \tag{3.4}$$

Using (2.6), we have

$$\nabla_a \det{}^{1/2}(I + \beta Q) = -2\beta\xi_d[HS(\beta)]^d{}_a \det{}^{1/2}(I + \beta Q), \tag{3.5}$$

$$\begin{aligned} \nabla_a S_{bc}(\beta) &= \beta S_{ba}(\beta)\xi^d[HS(\beta)]_{dc} - \beta S_{bd}(\beta)\xi^d[HS(\beta)]_{ac} \\ &\quad + \beta[HS(\beta)]_{ba}\xi^d S_{dc}(\beta) - \beta[HS(\beta)]_{bd}\xi^d S_{ac}(\beta). \end{aligned} \tag{3.6}$$

Combining these relations, we have

$$\nabla_a K_{bc}(\beta) = \det{}^{1/2}(I+\beta Q)\xi^d X_{abc;d}(\beta), \tag{3.7}$$

where

$$\begin{aligned} X_{abc;d}(\beta) = 2\beta[HS(\beta)]_{ad} S_{bc}(\beta) - \beta[HS(\beta)]_{bd} S_{ca}(\beta) - \beta[HS(\beta)]_{cd} S_{ab}(\beta) \\ + \beta S_{bd}(\beta)[HS(\beta)]_{ca} + \beta S_{cd}(\beta)[HS(\beta)]_{ba}. \end{aligned} \tag{3.8}$$

Then with the help of (3.7), it is easy to check that the following relations hold:

$$\nabla_{(a} K_{bc)}(\beta) = 0. \tag{3.9}$$

Therefore we have

$$\nabla_{(a} K^{(j)}_{bc)} = 0. \tag{3.10}$$

Proof of theorem 1. In terms of generating function, theorem 1 (2.18) can be written as follows:

$$K_{e(a}(\beta_1)\nabla^e K_{bc)}(\beta_2) - K_{e(a}(\beta_2)\nabla^e K_{bc)}(\beta_1) = 0. \tag{3.11}$$

Let

$$F_{abc}(\beta_1,\beta_2) := \frac{K_{ea}(\beta_1)\nabla^e K_{bc}(\beta_2)}{\det^{1/2}(I+\beta_1 Q)\det^{1/2}(I+\beta_2 Q)}. \tag{3.12}$$

(3.11) is equivalent to

$$F_{(abc)}(\beta_1,\beta_2) - F_{(abc)}(\beta_2,\beta_1) = 0. \tag{3.13}$$

Using the explicit form of $\nabla^e K_{bc}(\beta_2)$, we have

$$\begin{aligned} F_{abc}(\beta_1,\beta_2) &= \beta_2\xi^d S_{ea}(\beta_1) \\ &\quad\times \big(2[HS(\beta_2)]_{ed} S_{bc}(\beta_2) - [HS(\beta_2)]_{bd} S_c{}^e(\beta_2) - [HS(\beta_2)]_{cd} S^e{}_b(\beta_2) \\ &\quad + S_{bd}(\beta_2)[HS(\beta_2)]_c{}^e + S_{cd}(\beta_2)[HS(\beta_2)]_b{}^e\big) \\ &= \beta_2\xi^d\big(2[HS(\beta_1)S(\beta_2)]_{ad} S_{bc}(\beta_2) \\ &\quad - [HS(\beta_2)]_{bd}[S(\beta_1)S(\beta_2)]_{ca} - [HS(\beta_2)]_{cd}[S(\beta_1)S(\beta_2)]_{ab} \\ &\quad + S_{bd}(\beta_2)[HS(\beta_1)S(\beta_2)]_{ca} + S_{cd}(\beta_2)[HS(\beta_1)S(\beta_2)]_{ba}\big). \end{aligned} \tag{3.14}$$

Then

$$F_{(abc)}(\beta_1,\beta_2) = 2\beta_2\xi^d(S_{(bc}(\beta_2)[HS(\beta_1)S(\beta_2)]_{a)d} - [S(\beta_1)S(\beta_2)]_{(bc}[HS(\beta_2)]_{a)d}). \tag{3.15}$$

Note that

$$\beta_2 S(\beta_2) - \beta_1 S(\beta_1) = (\beta_2-\beta_1)S(\beta_1)S(\beta_2). \tag{3.16}$$

Then

$$\begin{aligned} F_{(abc)}(\beta_1,\beta_2) - F_{(abc)}(\beta_2,\beta_1) &= 2(\beta_2-\beta_1)\xi^d([S(\beta_1)S(\beta_2)]_{(bc}[HS(\beta_1)S(\beta_2)]_{a)d} \\ &\quad - [S(\beta_1)S(\beta_2)]_{(bc}[HS(\beta_1)S(\beta_2)]_{a)d}) \\ &= 0. \end{aligned}$$

This completes the proof of theorem 1. □

Let $\eta_a(\beta)$ be the generating function of $\eta^{(j)}_a$:

$$\eta_a(\beta) = \sum_{j=0}^{k-1} \eta^{(j)}_a \beta^j = K_{ab}(\beta)\xi^b. \tag{3.17}$$

Proof of theorem 2. In terms of the generating function (3.17), the theorem 2 is equivalent to

$$\mathcal{L}_{\eta(\beta)} h_{ab} = 0. \tag{3.18}$$

The left-hand side is

$$\mathcal{L}_{\eta(\beta)} h_{ab} = \eta^c(\beta)\nabla_c h_{ab} + h_{cb}\nabla_a \eta^c(\beta) + h_{ac}\nabla_b \eta^c(\beta). \tag{3.19}$$

Using (2.6), the first term on the right-hand side of (3.19) becomes

$$\eta^c(\beta)\nabla_c h_{ab} = \xi_b \eta_a(\beta) - \xi_a \eta_b(\beta). \tag{3.20}$$

Let us examine the second and third terms.

$$\begin{aligned} U_{ab}(\beta) :&= h_{cb}\nabla_a \eta^c(\beta) + h_{ac}\nabla_b \eta^c(\beta) \\ &= h_{cb}\nabla_a\big(K^c{}_d(\beta)\xi^d\big) + h_{ac}\nabla_b\big(K^c{}_d(\beta)\xi^d\big) \\ &= [K(\beta)H]_{db}\nabla_a\xi^d + [K(\beta)H]_{ad}\nabla_b\xi^d + \xi^d\big(h_{cb}\nabla_a K^c{}_d(\beta) + h_{ac}\nabla_b K^c{}_d(\beta)\big). \end{aligned} \tag{3.21}$$

Note that

$$[K(\beta)H]_{db}\nabla_a\xi^d + [K(\beta)H]_{ad}\nabla_b\xi^d = \mathcal{L}_\xi[K(\beta)H]_{ab} - \nabla_\xi[K(\beta)H]_{ab} = 0. \tag{3.22}$$

Here we have used (2.14) and (2.15).

Let

$$V_{ab}(\beta) := \frac{\xi^d h_{ac}\nabla_b K^c{}_d(\beta)}{\det^{1/2}(I+\beta Q)}. \tag{3.23}$$

Then

$$U_{ab}(\beta) = \det{}^{1/2}(I+\beta Q)(V_{ab}(\beta) - V_{ba}(\beta)) = 2\det{}^{1/2}(I+\beta Q)V_{[ab]}(\beta). \tag{3.24}$$

Using (3.7), we have

$$V_{ab}(\beta) = \beta\xi^d\xi^f\{[HS(\beta)]_{ad}[HS(\beta)]_{bf} - S_{df}[QS(\beta)]_{ab} + [QS(\beta)]_{ad}S_{bf}(\beta)\}, \tag{3.25}$$

$$2V_{[ab]}(\beta) = \beta\xi^d\xi^f\{[QS(\beta)]_{ad}S_{bf}(\beta) - S_{ad}(\beta)[QS(\beta)]_{bf}\}. \tag{3.26}$$

Note that

$$\beta QS(\beta) = I - S(\beta). \tag{3.27}$$

Then

$$2V_{[ab]}(\beta) = \beta\xi^d\xi^f\{g_{ad}S_{bf}(\beta) - S_{ad}(\beta)g_{bf}\} = \xi_a S_{bf}(\beta)\xi^f - \xi_b S_{ad}(\beta)\xi^d. \tag{3.28}$$

Therefore

$$U_{ab}(\beta) = \xi_a\eta_b(\beta) - \xi_b\eta_a(\beta). \tag{3.29}$$

Adding (3.20) and (3.29), we have

$$\mathcal{L}_{\eta(\beta)} h_{ab} = 0. \tag{3.30}$$

This completes the proof of theorem 2. □

The first relation of corollary is equivalent to

$$\mathcal{L}_{\eta^{(i)}} K^{(j)} = 0, \tag{3.31}$$

which immediately follows from theorem 2.

The second relation of corollary is equivalent to

$$\mathcal{L}_{\eta^{(i)}} \eta^{(j)} = 0. \tag{3.32}$$

Note that

$$\mathcal{L}_\xi \xi = [\xi, \xi] = 0, \tag{3.33}$$

$$\begin{aligned}\mathcal{L}_\xi \eta^{(j)a} &= \mathcal{L}_\xi \left(K^{(j)a}{}_b \xi^b\right) \\ &= \left(\mathcal{L}_\xi K^{(j)a}{}_b\right)\xi^b + K^{(j)a}{}_b(\mathcal{L}_\xi \xi^b) \\ &= 0.\end{aligned} \tag{3.34}$$

Here we have used (2.14) and (3.33). Then

$$\mathcal{L}_{\eta^{(j)}} \xi = [\eta^{(j)}, \xi] = -\mathcal{L}_\xi \eta^{(j)} = 0. \tag{3.35}$$

Now, using this relation and (3.31), we easily see that

$$\begin{aligned}\mathcal{L}_{\eta^{(i)}} \eta^{(j)a} &= \mathcal{L}_{\eta^{(i)}} \left(K^{(j)a}{}_b \xi^b\right) \\ &= \left(\mathcal{L}_{\eta^{(i)}} K^{(j)a}{}_b\right)\xi^b + K^{(j)a}{}_b(\mathcal{L}_{\eta^{(i)}} \xi^b) \\ &= 0.\end{aligned} \tag{3.36}$$

This completes the proof of corollary.

4. Separation of variables in the Hamilton–Jacobi equation

A geometric characterization of the separation of variables in the geodesic Hamilton–Jacobi equation was given by Benenti–Francaviglia [7] and Kalnins–Miller [8]. Here, we use the following result in [8].

Theorem. Suppose there exists a N-dimensional vector space $\mathcal{A}$ of rank-2 Killing tensors on D-dimensional space (M, g). Then the geodesic Hamilton–Jacobi equation has a separable coordinate system if and only if the following conditions hold[1]:

(i) $[A, B]_S = 0$ for each $A, B \in \mathcal{A}$.
(ii) There exist $(D - n)$-independent simultaneous eigenvectors $X^{(a)}$ for every $A \in \mathcal{A}$.
(iii) There exist n-independent commuting Killing vectors $Y^{(\alpha)}$.
(iv) $[A, Y^{(\alpha)}]_S = 0$ for each $A \in \mathcal{A}$.
(v) $N = (2D + n^2 - n)/2$.
(vi) $g(X^{(a)}, X^{(b)}) = 0$ if $1 \leqslant a < b \leqslant D - n$, and $g(X^{(a)}, Y^{(\alpha)}) = 0$ for $1 \leqslant a \leqslant D - n$, $D - n + 1 \leqslant \alpha \leqslant D$.

We assume that the Killing tensors $K^{(j)}$ and $K^{(ij)} = \eta^{(i)} \otimes \eta^{(j)} + \eta^{(j)} \otimes \eta^{(i)}$ given in section 2 form a basis for $\mathcal{A}$. Note that in the odd-dimensional case the last Killing Yano tensor $f^{(k-1)}$ is a Killing vector, and hence the corresponding Killing tensor $K^{(k-1)} \propto f^{(k-1)} f^{(k-1)}$ is reducible [5]. Then, it is easy to see that conditions (1)–(6) hold. Indeed, the relation $K^{(i)}K^{(j)} = K^{(j)}K^{(i)}$ implies that there exist simultaneous eigenvectors $X^{(a)}$ for $K^{(i)}$ satisfying conditions (2) and (6). Other conditions are direct consequences of theorem 1 and corollary.

5. Example

Finally, we describe the Kerr–NUT–de Sitter metric as an example, which was fully studied in [1–6, 13, 14]. The D-dimensional metric takes the form [13]:

[1] We put $n_2 = 0$ for theorem 4 in [8]. This condition is satisfied in the case of a positive definite metric g.

(a) $D = 2n$

$$g = \sum_{\mu=1}^{n} \frac{\mathrm{d}x_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} \mathrm{d}\psi_k \right)^2. \tag{5.1}$$

(b) $D = 2n + 1$

$$g = \sum_{\mu=1}^{n} \frac{\mathrm{d}x_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} \,\mathrm{d}\psi_k \right)^2 + S \left(\sum_{k=0}^{n} A^{(k)} \,\mathrm{d}\psi_k \right)^2. \tag{5.2}$$

The functions Q_μ are given by

$$Q_\mu = \frac{X_\mu}{U_\mu}, \qquad U_\mu = \prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^{n} \left(x_\mu^2 - x_\nu^2 \right), \tag{5.3}$$

where X_μ is a function depending only on x_μ and

$$A_\mu^{(k)} = \sum_{\substack{1 \leqslant \nu_1 < \cdots < \nu_k \leqslant n \\ (\nu_i \neq \mu)}} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2, \quad A^{(k)} = \sum_{1 \leqslant \nu_1 < \cdots < \nu_k \leqslant n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2, \quad S = \frac{c}{A^{(n)}} \tag{5.4}$$

with a constant c. The CKY tensor is written as [2]

$$h = \frac{1}{2} \sum_{k=0}^{n-1} \mathrm{d}A^{(k+1)} \wedge \mathrm{d}\psi_k \tag{5.5}$$

with the associated vector $\xi = \partial/\partial\psi_0$. Assumptions $(a1)$, $(a2)$ and $(a3)$ are clearly satisfied. The commuting Killing tensors $K^{(j)}$ and Killing vectors $\eta^{(j)}$ are calculated as [2, 3]

$$K^{(j)} = \sum_{\mu=1}^{n} A_\mu^{(j)} (\mathrm{e}^\mu\, \mathrm{e}^\mu + \mathrm{e}^{\mu+n}\, \mathrm{e}^{\mu+n}) + \epsilon A^{(j)} \mathrm{e}^{2n+1}\, \mathrm{e}^{2n+1}, \tag{5.6}$$

$$\eta^{(j)} = \frac{\partial}{\partial \psi_j}, \tag{5.7}$$

where $\epsilon = 0$ for $D = 2n$ and 1 for $D = 2n + 1$. The 1-forms $\{\mathrm{e}^\mu, \mathrm{e}^{\mu+n}, \mathrm{e}^{2n+1}\}$ are orthonormal bases defined by

$$\mathrm{e}^\mu = \frac{\mathrm{d}x_\mu}{\sqrt{Q_\mu}}, \qquad \mathrm{e}^{\mu+n} = \sqrt{Q_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} \,\mathrm{d}\psi_k \right), \qquad \mathrm{e}^{2n+1} = \sqrt{S} \left(\sum_{k=0}^{n} A^{(k)} \,\mathrm{d}\psi_k \right). \tag{5.8}$$

Note added. In the successive paper [15], we found that a single CKY tensor satisfying assumptions $(a1)$, $(a2)$ and $(a3)$ leads inevitably to the Kerr–NUT–de Sitter spacetime.

Acknowledgments

This work is supported by the 21 COE program ‘Construction of wide-angle mathematical basis focused on knots’. The work of YY is supported by the Grant-in Aid for Scientific Research (no. 19540304 and no. 19540098) from the Japan Ministry of Education. The work of TO is supported by the Grant-in Aid for Scientific Research (no. 18540285 and no. 19540304) from the Japan Ministry of Education.

Appendix A. Generating function of $K^{(j)}_{ab}$

In this appendix, we rederive the expression of the generating function of $K^{(j)}$ directly from the definition (2.13).

A.1. Auxiliary operators

It is convenient to introduce auxiliary fermionic creation/annihilation operators:

$$\bar\psi^a, \qquad \psi_a, \qquad a = 1, 2, \dots, D \tag{A.1}$$

such that

$$\{\psi_a, \psi_b\} = 0, \qquad \{\bar\psi^a, \bar\psi^b\} = 0, \qquad \{\psi_a, \bar\psi^b\} = \delta^b_a. \tag{A.2}$$

Also let

$$\bar\psi_a := g_{ab}\bar\psi^b, \qquad \psi^a := g^{ab}\psi_b. \tag{A.3}$$

$$\{\psi_a, \bar\psi_b\} = g_{ab}, \qquad \{\psi^a, \bar\psi^b\} = g^{ab}. \tag{A.4}$$

The Fock vacuum is defined by

$$\psi_a|0\rangle = 0, \qquad \langle 0|\bar\psi^a = 0, \qquad a = 1, 2, \dots, D, \tag{A.5}$$

with a normalization

$$\langle 0|0\rangle = 1. \tag{A.6}$$

With a 2-form h

$$h = \tfrac{1}{2} h_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b, \tag{A.7}$$

let us associate the following operators:

$$h_{\bar\psi} := \tfrac{1}{2} h_{ab}\bar\psi^a\bar\psi^b, \tag{A.8}$$

$$h_\psi := \tfrac{1}{2} h^{ab}\psi_a\psi_b. \tag{A.9}$$

Note that

$$(h_{\bar\psi})^j = \frac{1}{(2j)!} h^{(j)}_{a_1\dots a_{2j}} \bar\psi^{a_1}\cdots\bar\psi^{a_{2j}}. \tag{A.10}$$

$$h^{(j)}_{a_1\dots a_{2j}} = \langle 0|\psi_{a_{2j}}\cdots\psi_{a_1}(h_{\bar\psi})^j|0\rangle = (-1)^j\langle 0|\psi_{a_1}\cdots\psi_{a_{2j}}(h_{\bar\psi})^j|0\rangle. \tag{A.11}$$

A.2. The generating function of $A^{(j)}$

Let

$$A^{(j)} := \frac{1}{(2j)!(j!)^2}\left(h^{(j)}_{c_1\dots c_{2j}} h^{(j)c_1\dots c_{2j}}\right) = \frac{(2j)!}{(2^j j!)^2} h^{[a_1b_1}\cdots h^{a_jb_j]} h_{[a_1b_1}\cdots h_{a_jb_j]}. \tag{A.12}$$

$A^{(j)}$ is nontrivial for $j = 0, 1, \dots, [D/2]$.

Note that

$$\begin{aligned} A^{(j)} &= \frac{1}{(2j)!(j!)^2} h^{(j)}_{c_1\dots c_{2j}} h^{(j)c_1\dots c_{2j}} \\ &= \frac{1}{(2j)!(j!)^2} h^{(j)c_1\dots c_{2j}} \times (-1)^j \langle 0|\psi_{c_1}\cdots\psi_{c_{2j}}(h_{\bar\psi})^j|0\rangle \\ &= (-1)^j\langle 0|\frac{(h_\psi)^j}{j!}\frac{(h_{\bar\psi})^j}{j!}|0\rangle. \end{aligned} \tag{A.13}$$

Then we have

$$\sum_{j=0}^{[D/2]} A^{(j)}\beta^j = \langle 0|\,\mathrm{e}^{-\sqrt{\beta}h_\psi}\,\mathrm{e}^{\sqrt{\beta}h_{\bar\psi}}|0\rangle. \tag{A.14}$$

Let us introduce the vielbein

$$g_{ab} = \delta_{ij}e^i{}_a e^j{}_b. \tag{A.15}$$

(We assume the Euclidean signature.)

Let E be the matrix with elements

$$E^i{}_a = e^i{}_a. \tag{A.16}$$

Then

$$H^a{}_b = (E^{-1})^a{}_i \tilde{H}_{ij} E^j{}_b, \qquad \tilde{H}_{ij} = -\tilde{H}_{ji}. \tag{A.17}$$

Also let

$$\theta^i = e^i{}_a\psi^a, \qquad \bar\theta^i = e^i{}_a\bar\psi^a, \qquad i = 1, 2, \ldots, D. \tag{A.18}$$

Then we have $\theta_i = \theta^i$, $\bar\theta_i = \bar\theta^i$, and

$$\{\theta_i, \theta_j\} = 0, \qquad \{\bar\theta_i, \bar\theta_j\} = 0, \qquad \{\theta_i, \bar\theta_j\} = \delta_{ij}, \tag{A.19}$$

for $i, j = 1, 2, \ldots, D$. It is well known that any real antisymmetric matrix can be block diagonalized by some orthogonal matrix. Therefore, we can choose the vielbein such that $\tilde{H}$ has a block diagonal form and

$$h_\psi = \sum_{\mu=1}^{n}\lambda_\mu\theta_\mu\theta_{n+\mu}, \qquad h_{\bar\psi} = \sum_{\mu=1}^{n}\lambda_\mu\bar\theta_\mu\bar\theta_{n+\mu}, \tag{A.20}$$

for $n = [D/2]$. Here we assume that $\lambda_\mu \neq 0$. Note that

$$EQE^{-1} = \mathrm{diag}\big(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2, \lambda_1^2, \lambda_2^2, \ldots\big). \tag{A.21}$$

For odd D, the last diagonal entry equals zero.

Then

$$\begin{aligned}\langle 0|\,\mathrm{e}^{-\sqrt{\beta}h_\psi}\,\mathrm{e}^{\sqrt{\beta}h_{\bar\psi}}|0\rangle &= \langle 0|\prod_{\mu=1}^{n}(1-\sqrt{\beta}\lambda_\mu\theta_\mu\theta_{n+\mu})(1+\sqrt{\beta}\lambda_\mu\bar\theta_\mu\bar\theta_{n+\mu})|0\rangle \\ &= \prod_{\mu=1}^{n}\big(1+\beta\lambda_\mu^2\big) \\ &= \det{}^{1/2}(I+\beta Q).\end{aligned} \tag{A.22}$$

Here I is the $D\times D$ identity matrix.

We have the generating function of $A^{(j)}$:

$$\sum_{j=0}^{[D/2]} A^{(j)}\beta^j = \det{}^{1/2}(I+\beta Q) = \det(I+\sqrt{\beta}H) = \det(I-\sqrt{\beta}H). \tag{A.23}$$

A.3. Recursion relations for $K^{(j)}$

The Levi-Civita tensor satisfies

$$\varepsilon^{a_1\ldots a_r c_1\ldots c_{D-r}}\varepsilon_{b_1\ldots b_r c_1\ldots c_{D-r}} = r!(D-r)!\delta^{[a_1}_{b_1}\cdots\delta^{a_r]}_{b_r}. \tag{A.24}$$

Using (A.24), we can check that $K^{(j)}_{ab}$ has the following form:

$$K^{(j)}_{ab} = A^{(j)}g_{ab} + \frac{1}{(2j-1)!(j!)^2}h^{(j)}_{ac_1\ldots c_{2j-1}}h^{(j)c_1\ldots c_{2j-1}}{}_b. \tag{A.25}$$

Here $A^{(j)}$ is defined by (A.12).

It is possible to show that

$$\frac{1}{(2j-1)!(j!)^2}h^{(j)}_{ac_1\ldots c_{2j-1}}h^{(j)c_1\ldots c_{2j-1}}{}_b = h_a{}^c K^{(j-1)}_{cd}h^d{}_b. \tag{A.26}$$

In the matrix notation, $K^{(j)}$ satisfies the following recursion relation:

$$K^{(j)} = A^{(j)}I + HK^{(j-1)}H. \tag{A.27}$$

Therefore, we can see that $K^{(j)}$ commutes with H. Thus

$$K^{(j)} = A^{(j)}I - QK^{(j-1)}. \tag{A.28}$$

With the initial condition

$$K^{(0)} = I, \qquad K^{(0)}_{ab} = g_{ab}, \tag{A.29}$$

we easily find that

$$K^{(j)} = \sum_{l=0}^{j}(-1)^l A^{(j-l)}Q^l, \tag{A.30}$$

or

$$K^{(j)a}{}_b = \sum_{l=0}^{j}(-1)^l A^{(j-l)}(Q^l)^a{}_b. \tag{A.31}$$

We immediately see that

$$K^{(i)}K^{(j)} = K^{(j)}K^{(i)}. \tag{A.32}$$

Using (A.23), we can see that $K^{(k)} = 0$ for $k = [(D+1)/2]$. Indeed, by setting $\beta = -x^{-1}$,

$$\sum_{j=0}^{[D/2]}(-1)^j A^{(j)}x^{-j} = \det{}^{1/2}(I - x^{-1}Q) = x^{-D/2}\det{}^{1/2}(xI - Q). \tag{A.33}$$

For $D = 2k$,

$$\sum_{j=0}^{k}(-1)^{k-j}A^{(j)}x^{k-j} = (-1)^k\det{}^{1/2}(xI - Q). \tag{A.34}$$

If we set x to be an eigenvalue of Q, the RHS becomes zero. Therefore, we can see that

$$K^{(k)} = \sum_{l=0}^{k}(-1)^l A^{(k-l)}Q^l = 0, \qquad \text{for} \quad D = 2k. \tag{A.35}$$

Similarly, for $D = 2k-1$,

$$\sum_{j=0}^{k-1}(-1)^{k-j}A^{(j)}x^{k-j} = (-1)^k x^{1/2}\det{}^{1/2}(xI - Q). \tag{A.36}$$

Thus

$$K^{(k)} = \sum_{l=1}^{k} (-1)^l A^{(k-l)} Q^l = 0, \qquad \text{for} \quad D = 2k-1. \tag{A.37}$$

Also note that $A^{(j)} = 0$ for $j \geqslant [D/2] + 1$. Therefore the recursion relations (A.28) become trivial for $j \geqslant k+1$ and $K^{(j)}_{ab} = 0$ for $j \geqslant k$. $K^{(j)}$ can be written as (A.30) for all $j \geqslant 0$ but are nontrivial only for $j = 0, 1, \ldots, k-1$.

Using (A.30) and (A.23), we can see that the generating function of $K^{(j)}$ is

$$K(\beta) := \sum_{j=0}^{k-1} K^{(j)} \beta^j = \det{}^{1/2}(I + \beta Q)(I + \beta Q)^{-1}. \tag{A.38}$$

A.4. Proof of (A.26)

The LHS of (A.26) is

$$\begin{aligned}
&\frac{1}{(2j-1)!(j!)^2} h^{(j)}_{ac_1\ldots c_{2j-1}} h^{(j)c_1\ldots c_{2j-1}}{}_b \\
&\quad = \frac{1}{(2j-1)!(j!)^2} h^{(j)c_1\ldots c_{2j-1}}{}_b \times (-1)^j \langle 0|\psi_a \psi_{c_1} \cdots \psi_{c_{2j-1}} (h_{\bar\psi})^j |0\rangle \\
&\quad = \frac{(-1)^{j-1}}{(2j-1)!(j!)^2} h^{(j)c_1\ldots c_{2j-1}}{}_b \langle 0|\psi_{c_1} \cdots \psi_{c_{2j-1}} \psi_a (h_{\bar\psi})^j |0\rangle \\
&\quad = \frac{(-1)^{j-1}}{(2j)!(j!)^2} h^{(j)c_1\ldots c_{2j}} \langle 0|\psi_{c_1} \cdots \psi_{c_{2j}} \bar\psi_b \psi_a (h_{\bar\psi})^j |0\rangle \\
&\quad = (-1)^{j-1} \langle 0| \frac{(h_\psi)^j}{j!} \bar\psi_b \psi_a \frac{(h_{\bar\psi})^j}{j!} |0\rangle.
\end{aligned} \tag{A.39}$$

Then

$$\begin{aligned}
K^{(j)}_{ab} &= (-1)^j g_{ab} \langle 0| \frac{(h_\psi)^j}{j!} \frac{(h_{\bar\psi})^j}{j!} |0\rangle - (-1)^j \langle 0| \frac{(h_\psi)^j}{j!} \bar\psi_b \psi_a \frac{(h_{\bar\psi})^j}{j!} |0\rangle \\
&= (-1)^j \langle 0| \frac{(h_\psi)^j}{j!} [\{\psi_a, \bar\psi_b\} - \bar\psi_b \psi_a] \frac{(h_{\bar\psi})^j}{j!} |0\rangle \\
&= (-1)^j \langle 0| \frac{(h_\psi)^j}{j!} \psi_a \bar\psi_b \frac{(h_{\bar\psi})^j}{j!} |0\rangle.
\end{aligned} \tag{A.40}$$

Thus

$$K^{(j)}_{ab} = (-1)^j \langle 0| \frac{(h_\psi)^j}{j!} \psi_a \bar\psi_b \frac{(h_{\bar\psi})^j}{j!} |0\rangle. \tag{A.41}$$

Note that

$$[\psi_a, h_{\bar\psi}] = h_{aa'} \bar\psi^{a'}, \tag{A.42}$$

$$\psi_a (h_{\bar\psi})^j |0\rangle = j h_a{}^{a'} \bar\psi_{a'} (h_{\bar\psi})^{j-1} |0\rangle, \tag{A.43}$$

$$[h_\psi, \bar\psi_b] = \psi_{b'} h^{b'}{}_b, \tag{A.44}$$

$$\langle 0|(h_\psi)^j \bar\psi_b = j \langle 0|(h_\psi)^{j-1} \psi_{b'} h^{b'}{}_b. \tag{A.45}$$

Then

$$
\begin{aligned}
(\text{LHS of (A.26)}) &= \frac{1}{(2j-1)!(j!)^2} h^{(j)}_{ac_1\ldots c_{2j-1}} h^{(j)c_1\ldots c_{2j-1}}{}_b \\
&= (-1)^{j-1}\langle 0|\frac{(h_\psi)^j}{j!}\bar\psi_b\psi_a\frac{(h_{\bar\psi})^j}{j!}|0\rangle \\
&= h_a{}^{a'}(-1)^{j-1}\langle 0|\frac{(h_\psi)^{j-1}}{(j-1)!}\psi_{b'}\bar\psi_{a'}\frac{(h_{\bar\psi})^{j-1}}{(j-1)!}|0\rangle h^{b'}{}_b \\
&= h_a{}^{a'} K^{(j-1)}_{a'b'} h^{b'}{}_b \\
&= (\text{RHS of (A.26)}).
\end{aligned}
\tag{A.46}
$$

This completes the proof of (A.26).

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